On the convergence of message passing computation of harmonic influence in social networks

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Abstract—The harmonic influence is a measure of node influence in social networks that quantifies the ability of a leader node to alter the average opinion of the network, acting against an adversary field node. The definition of harmonic influence assumes linear interactions between the nodes described by an undirected weighted graph; its computation is equivalent to solve a discrete Dirichlet problem associated to a grounded Laplacian for every node. This measure has been recently studied, under slightly more restrictive assumptions, by Vassio et al., IEEE Trans. Control Netw. Syst., 2014, who proposed a distributed message passing algorithm that concurrently computes the harmonic influence of all nodes. In this paper, we provide a convergence analysis for this algorithm, which largely extends upon previous results: we prove that the algorithm converges asymptotically, under the only assumption of the interaction Laplacian being symmetric. However, the convergence value does not in general coincide with the harmonic influence: by simulations, we show that when the network has a larger number of cycles, the algorithm becomes slower and less accurate, but nevertheless provides a useful approximation. Simulations also indicate that the symmetry condition is not necessary for convergence and that performance scales very well in the number of nodes of the graph.

Index Terms—Distributed algorithm, message passing, opinion dynamics, social networks.

1 Introduction

In the study of networks and dynamical processes therein, one important issue is the identification of the most influential nodes, i.e. those with the higher ability to drive the others towards a desired state. The issue depends on the process and the control objective: consequently, it has been addressed in several contexts, from the seminal paper [1] on maximizing the spreading of influence, to several leader selection problems recently considered, such as [2], [3], [4], [5], [6], [7], [8].

In this work, we formulate this problem in the context of social influence networks. Following a consolidated research line [9], [10], [11], we postulate that the opinions of the nodes follow a linear dynamics with fixed confidence weights. We assume that a *leader* node has to compete against a given adversary *field* node in order to win the opinions of the other nodes. Under these assumptions, the fixed point of the opinion dynamics is the solution of a Dirichlet problem for the Laplacian of the graph, where the leader and the field fix the boundary constraints.

Assuming without loss of generality that the leader has opinion one and the external field has opinion zero, we define the *harmonic influence* of the leader as the sum of the asymptotic opinions reached by the agents in the social network. The influence of a node is the influence obtained if that node was the leader. This quantity was implicitly defined in [5] and named *Harmonic Influence Centrality* in [6].

By its definition, the harmonic influence of each node can be computed exactly by solving an array of n linear systems defined by the Laplacian of the graph, "grounded" in each of the n nodes and the field node [12]. This straightforward approach, used in [5], has some drawbacks. Firstly, global knowledge of the graph and update matrix is required by most solution methods, with the exception of some distributed (i.e. non-global) methods like [13] and [14]. Secondly, solving n systems is computationally expensive, even if one can resort to state-of-the-art algorithms that are tailored to Laplacian systems: these methods can solve each system in a time proportional to the number of edges but are not distributed [15]. Moreover, since the n systems are obtained by grounding the same original Laplacian, solving them separately is wastefully redundant. Alternatively, the harmonic influence can be computed iteratively by simply running the linear opinion dynamics n times, one for each possible leader node. Despite being distributed, this method remains not scalable.

In order to overcome this scalability issue, paper [6] proposed a *Message Passing Algorithm* (MPA) able to concurrently compute the influence of all nodes. This algorithm is *distributed*, that is, does not require any global knowledge of the graph or of the parameters of the opinion dynamics: moreover, it computes the harmonic influence of all nodes at the same time. The algorithm is based on the crucial assumption that the graph is undirected, that is, interactions are reciprocal. If the graph is an effective tree (that is, if it is connected and removing the field node makes it a forest), then the algorithm computes the nodes' influence in a number of steps equal to the diameter of the graph. The

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algorithm thus scales very nicely in the size of the graph. If the graph is connected and the Laplacian matrix is symmetric, then the algorithm converges asymptotically. Our main contribution is indeed the proof of this convergence result, which subsumes all previously available results for unweighted regular graphs [6] and for unicyclic graphs [16]. It must be stressed that in general the algorithm, even though it converges, does not converge to the exact values of the influence: exactness is only guaranteed on effective trees. We complement our mathematical analysis with extended simulations on synthetic random graphs, from which we draw three relevant observations: (1) When the number of cycles increases, the algorithm becomes slower and less accurate, but nevertheless provides a useful approximation of the harmonic influence; (2) When the number of nodes increases, the performance of the algorithm is only marginally affected: thus the algorithm scales very well to large graphs; (3) For the algorithm to converge, the symmetry of the Laplacian is unnecessary.

Further relations with the literature

Our paper contributes to the literature on message passing algorithms, by providing an interesting example of algorithm that converges on *any* graph. On the contrary, proofs of convergence of message passing algorithms are often limited to tree graphs or to locally-tree-like graphs [17].

In this field, a closely related paper is [13], which reformulates the problem of solving a linear systems $A\mathbf{x} = \mathbf{b}$, where the matrix A is full rank and symmetric, into a probabilistic inference problem. Then, it develops a Gaussian belief propagation method that involves two kinds of messages. The authors prove that under suitable conditions the algorithm converges to the exact solution. On trees, the algorithm coincides with the direct Gaussian elimination method.

Our work also shares some ideas with [18], which proposes a *consensus propagation* protocol based on two kinds of messages to solve the consensus problem: one contains a partial estimate of the consensus value and the other contains the number of nodes involved in such partial estimate. A suitable attenuation parameter makes the protocol [18] convergent on general graphs.

Furthermore, if we interpret the harmonic influence as a kind of centrality measure, then we should mention that some literature has looked at distributed algorithms to compute other centrality measures, such as closeness [19], betweenness [20], and eigenvector centrality or PageRank [21], [22].

Paper Structure

Section 2 defines the harmonic influence and Section 3 describes our Message Passing Algorithm for its concurrent and distributed computation, whereas the technical proofs of convergence are given in Section 4. Simulations are presented in Section 5 and Section 6 concludes the paper.

Notation

The set of real and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. Vectors are denoted with boldface letters and matrices with capital letters. The vectors $\mathbf{0}$ and $\mathbf{1}$ denote respectively the all-zero and all-one vectors of appropriate dimension. The symbol $\mathbb{I}$ denotes any identity matrix with appropriate dimension. The symbol $\preccurlyeq$ denotes entry-wise $\leq$ for vectors and matrices. The symbol $\prec$ is used if the entry-wise inequality is strict for at least one entry. Given a matrix Q, $Q^\top$ denotes its transpose, Q^{-1} its inverse and $\rho(Q)$ its spectral radius, i.e. the maximum absolute value of the eigenvalues of Q. If $\rho(Q) < 1$, Q is termed "Schur stable". Given a vector $\mathbf{v}$, $\mathrm{Diag}(\mathbf{v})$ is the square diagonal matrix with the entries of $\mathbf{v}$ on the main diagonal. The cardinality of the set S is denoted by $|S|$. The symbol $\subset$ is used for strict subsets; $\subseteq$ for generic subsets. Given the matrix $Q \in \mathbb{R}^{S\times S}$ and two subsets $T, T' \subseteq S$, $Q_{T,T'}$ is the sub-matrix of Q containing the rows and columns corresponding to T and T', respectively. A non-negative matrix $Q \in \mathbb{R}_+^{S\times S}$ is said to be stochastic, sub-stochastic and strictly sub-stochastic if $Q\mathbf{1} = \mathbf{1}$, $Q\mathbf{1} \preccurlyeq \mathbf{1}$ and $Q\mathbf{1} \prec \mathbf{1}$, respectively.

Let $\mathcal{G} = (V, E)$ be a graph where V is the set of vertices and E is the set of edges, which are unordered pairs of vertices. We will use the terms *node*, *vertex* and *agent* interchangably. The set $N_v = \{w \in V : \{v, w\} \in E\}$ contains the neighbors of v in $\mathcal{G}$; the degree of v is $d_v = |N_v|$. A *leaf* is a node of degree one. The graph $\mathcal{G}' = (V', E')$ is a subgraph of $\mathcal{G} = (V, E)$ if $V' \subset V$ and $E' \subset E$. If $\mathcal{G}'$ contains all edges of $\mathcal{G}$ that join two vertices in V', then $\mathcal{G}'$ is said to be the subgraph *induced by* V' and is denoted by $\mathcal{G}[V']$. A *path* is a graph $\mathcal{P} = (V_\mathcal{P}, E_\mathcal{P})$ of the form:

$$\begin{aligned}
&V_\mathcal{P} = \{u_0, u_1, \ldots, u_\ell\}\,,\\
&E_\mathcal{P} = \{\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{\ell-1}, u_\ell\}\}\,.
\end{aligned}$$

The vertices u_0 and u_ℓ are the *endvertices* of $\mathcal{P}$ and ℓ is the length of $\mathcal{P}$ [23]. Given a path of length $\ell \geq 2$, we term *cycle* the graph $(V_\mathcal{P}, E_\mathcal{P} \cup \{\{u_0, u_\ell\}\})$. A graph $\mathcal{G} = (V, E)$ is *connected* if for any pair of nodes $v, w \in V$ it admits a path with endvertices v, w as a subgraph. If $\mathcal{G}$ is connected, the *distance* between v and w is the minimal length of the path subgraphs with endvertices v, w while the *diameter* of $\mathcal{G}$ is the maximum distance between pairs of nodes.

2 The Harmonic Influence

Consider a simple weighted graph $\mathcal{G} = (I, E, C)$ with node set $I = \{\mathfrak{f}, 1, 2, \ldots, n\}$ of cardinality $n+1$ where $\mathfrak{f}$ is a special node called *field*. The edge set E contains unordered pairs of nodes and the non-negative weight matrix $C \in \mathbb{R}_+^{I\times I}$ is such that C_{ij} and C_{ji} are both non-zero if and only if $\{i, j\} \in E$. Note that C needs not to be symmetric, but its zeros are symmetric and its main diagonal is null. We also introduce the diagonal matrix $D = \mathrm{Diag}(C\mathbf{1})$ and the *Laplacian* matrix $L = D - C$.

We define the *harmonic influence* of the nodes in $I \setminus \{\mathfrak{f}\}$ as follows. Given a node $\ell \neq \mathfrak{f}$ where ℓ stands for *leader*, we denote the set of remaining nodes by $R^\ell := I \setminus \{\mathfrak{f}, \ell\}$ and consider the Laplacian system with boundary conditions (Dirichlet problem):

$$\begin{cases}
(L\,\mathbf{x})_{R^\ell} = \mathbf{0}\\
x_\ell = 1\\
x_\mathfrak{f} = 0\,.
\end{cases} \tag{1}$$

The *harmonic influence* of ℓ is the sum of entries of the vector $\mathbf{x}$ solution of (1), that is,

$$H(\ell) := \mathbf{1}^\top \mathbf{x}\,. \tag{2}$$

The following result guarantees that harmonic influence is well defined for connected graphs.

Lemma 1. *Assume the graph $\mathcal{G} = (I, E, C)$ to be connected. Then, for any $\ell \in I \setminus \{\mathfrak{f}\}$, the Laplacian system* (1) *admits a unique solution and $H(\ell)$ can be computed as:*

$$H(\ell) = 1 + \mathbf{1}^\top (L_{R^\ell,R^\ell})^{-1}\, C_{R^\ell,\{\ell\}}\,.$$

Moreover, $H(\ell) \in [1, n]$.

Proof. We rewrite $(L\,\mathbf{x})_{R^\ell} = \mathbf{0}$ as:

$$L_{R^\ell,R^\ell}\mathbf{x}_{R^\ell} + L_{R^\ell,\{\ell\}} x_\ell + L_{R^\ell,\{\mathfrak{f}\}} x_{\mathfrak{f}} = \mathbf{0}\,,$$

and obtain:

$$L_{R^\ell,R^\ell}\mathbf{x}_{R^\ell} = C_{R^\ell,\{\ell\}}\,,$$

using $L_{R^\ell,\{\ell\}} = -C_{R^\ell,\{\ell\}}$ and the boundary conditions. To prove that L_{R^ℓ,R^ℓ} is invertible we can equivalently work with $D^{-1}_{R^\ell,R^\ell} L_{R^\ell,R^\ell}$, because the graph $\mathcal{G}$ is connected and the matrix D as well as any of its principal sub-matrices are invertible. We have:

$$D^{-1}_{R^\ell,R^\ell} L_{R^\ell,R^\ell} = \mathbb{I} - D^{-1}_{R^\ell,R^\ell} C_{R^\ell,R^\ell} = \mathbb{I} - (D^{-1}C)_{R^\ell,R^\ell}\,,$$

thanks to the fact that D is diagonal. The matrix $D^{-1}C$ is stochastic and the graph $\mathcal{G}$ is connected, thus the principal sub-matrix $(D^{-1}C)_{R^\ell,R^\ell}$ is strictly sub-stochastic and Schur stable [24, Lemma 5]. Therefore the matrix $\mathbb{I} - (D^{-1}C)_{R^\ell,R^\ell}$ is invertible.

Finally, note that $x_i \in [0, 1]$ for every $i \in R^\ell$ because they solve a linear Laplacian system with boundary conditions in $[0, 1]$, so $H(\ell) \in [1, n]$. □

Before describing our approach to compute H, in the rest of this section we offer an interpretation of the *harmonic influence* based on a linear opinion dynamic model in an undirected connected network with two stubborn leaders.

2.1 Opinion dynamics interpretation

Assume that the weighted graph $\mathcal{G} = (I, E, C)$ is connected and represents a social network where agents are endowed with a scalar opinion $x_i(t)$ updated at discrete time steps $t \in \mathbb{N}$. The node $\mathfrak{f}$ is a stubborn *leader* with null opinion, i.e. $x_{\mathfrak{f}}(t) = 0$ for every $t \geq 0$. Also the agent $\ell \neq \mathfrak{f}$ is a stubborn *leader*, with conflicting opinion $x_\ell(t) = 1$ for every $t \geq 0$. The remaining *regular* agents in $R^\ell = I \setminus \{\mathfrak{f}, \ell\}$ have initial opinion $x_i(0) \in \mathbb{R}$. At each step, they update their opinion to a convex combination of the opinion of their neighbors:

$$x_i(t+1) = \textstyle\sum_{j\in I} Q_{ij}\, x_j(t) \qquad \forall t \geq 0\,, \tag{3}$$

where Q_{ij} is an element of the stochastic matrix $Q = D^{-1}C$ and represents how much agent i trusts agent j. The vector $\mathbf{x}(t) \in \mathbb{R}^I$ that stacks the agents' opinion converges to the solution of the Laplacian system.

Lemma 2. *Assume the graph $\mathcal{G} = (I, E, C)$ is connected with $n \geq 1$. The vector $\mathbf{x}(t)$ converges to the solution of* (1).

Proof. The statement is trivial for the agents $\mathfrak{f}$ and ℓ. The update rule of the regular agent, in compact form, is:

$$\mathbf{x}_{R^\ell}(t+1) = Q_{R^\ell,R^\ell}\mathbf{x}_{R^\ell}(t) + Q_{R^\ell,\{\ell\}}\,,$$

which implies:

$$\mathbf{x}_{R^\ell}(t) = (Q_{R^\ell,R^\ell})^t\, \mathbf{x}_{R^\ell}(0) + \textstyle\sum_{i=0}^{t-1} (Q_{R^\ell,R^\ell})^i\, Q_{R^\ell,\{\ell\}}\,.$$

As we argued in the proof of Lemma 1, the matrix $Q_{R^\ell,R^\ell} = (D^{-1}C)_{R^\ell,R^\ell}$ is Schur stable. Hence:

$$\begin{aligned}\lim_{t\to\infty} \mathbf{x}_{R^\ell}(t) &= \textstyle\sum_{i=0}^{\infty} (Q_{R^\ell,R^\ell})^i\, Q_{R^\ell,\{\ell\}}\\ &= (\mathbb{I} - Q_{R^\ell,R^\ell})^{-1}\, Q_{R^\ell,\{\ell\}}\,.\end{aligned}$$

If we multiply for $D^{-1}_{R^\ell,R^\ell} D_{R^\ell,R^\ell}$ between the two terms, we finally obtain $\lim_{t\to\infty} \mathbf{x}_{R^\ell}(t) = (L_{R^\ell,R^\ell})^{-1}\, C_{R^\ell,\{\ell\}}\,.$ □

Lemma 2 implies that the harmonic influence of $\ell \neq \mathfrak{f}$ is the sum of the asymptotic agents' opinion in the undirected weighted connected network $\mathcal{G} = (I, E, C)$ subject to a linear opinion dynamic model with two stubborn leaders, ℓ itself with opinion 1 and $\mathfrak{f}$ with opinion 0:

$$H(\ell) = \lim_{t\to\infty} \mathbf{1}^\top \mathbf{x}(t)\,. \tag{4}$$

The vector $\mathbf{x}(t)$ does not converge to a consensus. Observe however that if the leader ℓ was not present and every agent in $\{1, \ldots, n\}$ still updated his opinion according to (3), then consensus would be reached with $\mathbf{x}(t) \to \mathbf{0}$ [25, Thm. 13]. Therefore, we interpret the leader $\mathfrak{f}$ as the one originating a null opinion field in the social network. The harmonic influence $H(\ell)$ measures how effective ℓ is in diffusing a different opinion. Following (4), H can be computed by running n dynamics (3), one for each possible leader ℓ. This approach being non scalable in n motivates the scalable distributed method that is studied in the rest of this paper.

3 Distributed computation of the influence

We present a Message Passing Algorithm (MPA) able to compute concurrently and in a distributed way the *harmonic influence* of every non-field node of a connected graph.

Following the definition, the computation of the *harmonic influence* of every node $\ell \neq \mathfrak{f}$ requires the solution of n Laplacian systems like (1). The plain application of Lemma 1 requires global knowledge of the graph and of the Laplacian matrix L. Moreover, it does not exploit the apparent redundancies between the n systems, as the Laplacian matrix L does not change while different principal sub-matrices are used. The paper [6] proposed a different, more scalable, approach, that uses a MPA: in the following we recall and extend its definition.

Consider the simple weighted graph $\mathcal{G} = (I, E, C)$ and let $t \in \{0, 1, \ldots\}$ be an iteration counter. At each step, every node i sends to its neighbors j two messages:

$$W^{i\to j}(t) \in [0, 1]\,, \qquad H^{i\to j}(t) \in \mathbb{R}_+\,.$$

The field node $\mathfrak{f}$ sends null messages:

$$W^{\mathfrak{f}\to j}(t) = 0\,, \quad H^{\mathfrak{f}\to j}(t) = 0\,, \quad \forall j \in N_{\mathfrak{f}}\,, \quad \forall t \geq 0\,,$$

whereas any other node $i \neq \mathfrak{f}$ sends the initial messages:

$$W^{i\to j}(0) = 1\,, \quad H^{i\to j}(0) = 1\,, \quad \forall j \in N_i\,.$$

and then synchronously updates the messages sent to his neighbor j following the rules:

$$W^{i\to j}(t+1) = \left(1 + \sum_{k\in N_i^j} \frac{C_{ik}}{C_{ij}} \Big(1 - W^{k\to i}(t)\Big)\right)^{-1} \tag{5}$$

$$H^{i\to j}(t+1) = 1 + \sum_{k\in N_i^j} W^{k\to i}(t)\, H^{k\to i}(t)\,, \tag{6}$$

where $N_i^j := N_i \setminus \{j\}$ is the set of neighbors of i except the one to which the message is sent. At any time, any node ℓ in $I \setminus \{\mathfrak{f}\}$ can compute an approximation of its *harmonic influence* $H(\ell)$ using the incoming messages:

$$H^\ell(t) = 1 + \sum_{i\in N_\ell} W^{i\to\ell}(t)\, H^{i\to\ell}(t)\,.$$

As observed in [6], the MPA converges to H in a finite time if the graph $\mathcal{G}$ is a tree. Actually, this property is valid for a slightly larger class of graphs defined as follows.

Definition. *The graph $\mathcal{G} = (I, E, C)$ is an* effective tree *if it is connected and the induced subgraph $\mathcal{G}[I \setminus \{\mathfrak{f}\}]$ is a forest.*

Basically, an *effective tree* is any connected graph $\mathcal{G}$ that after the removal of the field node $\mathfrak{f}$ is a forest. The location of the field node $\mathfrak{f}$ allows *effectively* the same kind of computation done on tree graphs.

Proposition 3. *If the graph $\mathcal{G} = (I, E, C)$ is an effective tree and δ is its diameter, then:*

$$H^\ell(t) = H(\ell) \qquad \forall t \geq \delta - 1, \quad \forall \ell \in I \setminus \{\mathfrak{f}\}\,.$$

Proposition 3 will be proved in the next section by showing that messages converge after a finite number of steps and constructing the solution of the Laplacian system for given ℓ. In an effective tree the convergence values of the messages $W^{i\to j}(t)$ and $H^{i\to j}(t)$ have an exact interpretation. Although the correct interpretation will be evident in the proof, we anticipate it here (see also Figure 1):

- $W^{i\to j}(\infty)$ is the value x_i in the Laplacian system (1) where the leader ℓ is actually j ;
- $H^{i\to j}(\infty)$ is the harmonic influence $H(i)$ of the node i in the graph obtained from $\mathcal{G}$ by removing the edge $\{i, j\}$ and adding a new edge $\{i, \mathfrak{f}\}$.

Our main result guarantees the asymptotic convergence of the MPA on connected graphs $\mathcal{G} = (I, E, C)$ with symmetric weight matrix C.

Theorem 4 (Convergence)**.** *The MPA converges on any connected graph $\mathcal{G} = (I, E, C)$ with symmetric weight matrix C.*

The proof of Theorem 4, which is detailed in the next section, is based on the following two key ideas:

1) construct a directed graph of relations between messages (called message digraph and denoted by $\mathcal{M}_\mathcal{G}$) and study its connectivity properties, which descend from those of $\mathcal{G}$;
2) define a generalisation of the MPA on directed graphs and prove its convergence.

To complete the proof, these two ideas are combined by recognising that the MPA algorithm induces an MPA-like dynamics on its message digraph.

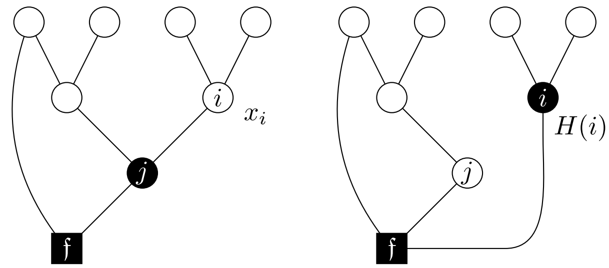

Fig. 1. Two graphs with the node $\mathfrak{f}$ marked by a black square and the leader ℓ marked by a black circle: in the left graph the leader is the node j, in the right one it is the node i. Let the effective tree on the left be $\mathcal{G}$: the message $W^{i\to j}(\infty)$ is the value of x_i in the Laplacian system (1) where the leader ℓ is the node j. The graph on the right is obtained from $\mathcal{G}$ by substituting the edge $\{i, j\}$ with the edge $\{i, \mathfrak{f}\}$ and is also an effective tree. The message $H^{i\to j}(\infty)$ is the harmonic influence $H(i)$ of i in this modified graph.

In comparison with Proposition 3, Theorem 4 guarantees that the MPA converges even if the connected graph $\mathcal{G}$ is not an effective tree, provided C is symmetric. However, convergence is asymptotical (not in finite time) and the limit values do not in general provide the exact values of the harmonic influence (that is, $H^\ell(\infty) \neq H(\ell)$). We shall explore the issues of convergence time and of asymptotical error by simulations in Section 5. In the same section we will conjecture that the MPA also converges for non symmetric matrices C.

3.1 Relation with the paper [6]

The MPA was originally proposed by [6] to compute the harmonic influence in graphs $\mathcal{G} = (I, E, C)$ with symmetric matrix C. Those graphs can be interpreted as electrical networks: each edge $\{i, j\}$ has conductance $C_{ij} = C_{ji}$ and the field node $\mathfrak{f}$ is a reference with null electrical potential. The harmonic influence $H(\ell)$ coincides with the sum of the nodes' electrical potential in the network where the potential of ℓ is held at one by an external battery. The set of $n-1$ independent node equations obtained using Ohm law and Kirchhoff's current law coincide with the Laplacian system (1). See also [26] about the connection between social and electrical networks. On (effective) trees, computations based on the concept of effective resistance are exact and have a recursive structure, which has inspired the design of the MPA [6].

Proposition 3 shows that the MPA can be extended to graphs with non-symmetric matrix C. More precisely, we just assume that C has null diagonal and symmetric pattern of zeros. Thus, the proposition distinguishes between C_{ij} and C_{ji} and guarantees that the update rule (5) is actually the correct extension of the rule in [6]. Theorem 4 proves the convergence of the MPA on every weighted connected graph where C is symmetric and extends the result in [6] about unweighted connected regular graphs.

4 Convergence proofs

This section is devoted to the proofs of Proposition 3 and Theorem 4. The proof of Proposition 3, given in Section 4.1, is direct and based on a triangularization procedure allowed by the acyclic structure of the system. The rest of the section is devoted to the proof of Theorem 4, which proceeds in three steps that develop the key ideas highlighted above. In Section 4.2 we define the message digraph $\mathcal{M}_{\mathcal{G}}$ and describe its connectivity properties. In Section 4.3 we define the non-linear dynamics (7)-(8) on directed graphs, which is a generalization of the MPA algorithm, and we prove its convergence. This convergence argument proceeds by distinguishing between graphs with different topologies: we first study acyclic graphs and strongly connected graphs, and then combine the results to obtain convergence on generally connected graphs. Finally, in Section 4.4 we recognise that the MPA can be mapped into a special case of this dynamic and thus prove its convergence. Instrumental to this identification is the presence in (7)-(8) of time-dependent terms that allow us to accommodate the messages originating from the field node $\mathfrak{f}$. At the very end of the section, we shall observe that our proof of convergence of the messages $W^{i\to j}(t)$ does not need the symmetry of C.

4.1 Convergence on effective trees

Proof of Proposition 3. First we prove that the messages converge in finite time and then we prove that the convergence values lead to the computation of the exact harmonic influence. Let the set $\vec{E} \subseteq I \times I$ contain all the *ordered* pairs of vertices of I that share an edge in $\mathcal{G}$:

$$\vec{E} := \{(j,i) : \{i,j\} \in E\}$$

We endow each element of $\vec{E}$ with a non-negative "order" integer $o_{(j,i)}$ whose value is given by the following recursive construction independent from ℓ:

$$\begin{cases} o_{(j,i)} = 0 & \text{if } i = \mathfrak{f} \text{ or } N_i^j = \emptyset\,, \\ o_{(j,i)} = 1 + \max_{k\in N_i^j} o_{(i,k)} & \text{otherwise}\,, \end{cases}$$

where $N_i^j = N_i \setminus \{j\}$. Basically these integers are assigned starting from the leaves of $\mathcal{G}$ and the node $\mathfrak{f}$ and proceding sequentially. There exists a unique and unambiguous way to assign these integers because $\mathcal{G}$ is an effective tree: any cycles in $\mathcal{G}$ contains the node $\mathfrak{f}$. It is easy to see that $\max_{(j,i)\in\vec{E}} o_{(j,i)} = \delta - 1$, where δ is the diameter of $\mathcal{G}$, and by induction that:

$$W^{i\to j}(t) = W^{i\to j}(o_{(j,i)})\,, \qquad H^{i\to j}(t) = H^{i\to j}(o_{(j,i)})\,,$$

for every $t \geq o_{(j,i)}$ so the messages converge in finite time.

Now, fix the node ℓ and let $\mathbf{x}$ be the solution of (1). We introduce a second iterative construction that proceeds from the leaves and field node towards the node ℓ and whose actual goal is to produce a triangularization of the Laplacian matrix and thus compute $\mathbf{x}$ and the sum of $\mathbf{x}$.

For its initial step we consider the field node $\mathfrak{f}$ and the leaves separately. First, consider the former and all its neighbors in $N_{\mathfrak{f}}$ and notice that:

$$x_{\mathfrak{f}} = 0 = W^{\mathfrak{f}\to j}(o_{(j,\mathfrak{f})})x_j\,,$$

where $j \in N_{\mathfrak{f}}$ because $W^{\mathfrak{f}\to j}(o_{(j,\mathfrak{f})}) = W^{i\to j}(0) = 0$. The contribution of $\mathfrak{f}$ to the harmonic influence of ℓ is null and we rewrite it as $H^{\mathfrak{f}\to j}(o_{(j,\mathfrak{f})})x_{\mathfrak{f}}$ with $H^{\mathfrak{f}\to j}(o_{(j,\mathfrak{f})}) = 0$. Second, consider any leaf node $i \notin \{\ell, \mathfrak{f}\}$ and let j be its unique neighbor, i.e. $N_i = \{j\}$. The equation $(L\mathbf{x})_{\{i\}} = 0$ is $C_{ij}(x_i - x_j) = 0$ and we rewrite it as:

$$x_i = x_j = W^{i\to j}(o_{(j,i)})x_j\,,$$

because $W^{i\to j}(o_{(j,i)}) = 1$. The contribution of x_i to the harmonic influence of ℓ can be expressed as $H^{i\to j}(o_{(j,i)})x_i$ with coefficient $H^{i\to j}(o_{(j,i)}) = 1$.

To describe the iterative step, consider a node $i \neq \ell$ such that the equation of all but a neighbor j have been already rewritten as $x_k = W^{k\to i}(o_{(i,k)})x_i$. Assume the number $H^{k\to i}(o_{(i,k)})x_k$ is the contribution to the harmonic influence of ℓ coming from node k and those nodes connected to k for which the equations have been already rewritten. We rewrite the equation $(L\mathbf{x})_{\{i\}} = 0$ as follows:

$$\sum_{k\in N_i} C_{ik}(x_i - x_k) = 0$$
$$\sum_{k\in N_i^j} C_{ik}\big(1 - W^{k\to i}(o_{(i,k)})\big)x_i + C_{ij}x_i = C_{ij}x_j$$
$$x_i = \frac{C_{ij}}{C_{ij} + \sum_{k\in N_i^j} C_{ik}\big(1 - W^{k\to i}(o_{(i,k)})\big)}x_j$$

and then recognize that $x_i = W^{i\to j}(o_{(j,i)})x_j$. We stress that this rewritings are unambiguous because $\mathcal{G}$ is an effective tree. The contribution to the harmonic influence of ℓ by node i and those nodes connected to i for which the corresponding equations have been already rewritten is $H^{i\to j}(o_{(j,i)})x_i$ where the coefficient satisfies:

$$H^{i\to j}(o_{(j,i)}) = 1 + \sum_{k\in N_i^j} W^{k\to i}(o_{(i,k)})H^{k\to i}(o_{(i,k)})\,.$$

The iterative procedure repeats until all the equations have been rewritten, except that corresponding to node ℓ for which $x_\ell = 1$. The harmonic influence of ℓ can be finally computed summing the contribution coming from all branches of the graph stemming from ℓ:

$$H^\ell(\max_{i\in N_\ell} o_{(\ell,i)}) = 1 + \sum_{i\in N_\ell} W^{i\to\ell}(o_{(\ell,i)})\, H^{i\to\ell}(o_{(\ell,i)}).$$

Making explicit all the intermediate relations:

$$H^\ell(\max_{i\in N_\ell} o_{(\ell,i)}) = \sum_{i\in I} x_i = H(\ell)\,.$$

The thesis follows because ℓ is arbitrary. □

4.2 The message digraph $\mathcal{M}_{\mathcal{G}}$ and its topology

First, we introduce *directed* graphs and the related notation. Then, we define the *message digraph* $\mathcal{M}_{\mathcal{G}}$ associated to the graph $\mathcal{G} = (I, E, C)$ and prove a topological property valid if $\mathcal{G}$ is connected.

A *directed* graph or *digraph* is a pair $\mathcal{D} = (V, \Phi)$ where V is the set of vertices and $\Phi \subseteq V \times V$ is the set of *arcs*, that are ordered pairs of vertices. The sub-digraph induced by $U \subseteq V$ is $\mathcal{D}[U] = (U, \Phi \cap U \times U)$. A node v is a *sink* if $(v, w) \notin \Phi$ for any $w \in V$. An arc of the form (v, v) is a *self-loop*. A *walk from v to w* on the digraph $\mathcal{D}$, of length l, is an ordered list of nodes $(u_0, u_i, \ldots, u_l)$ such that:

(i) $u_0 = v$ and $u_l = w$;
(ii) $(u_{i-1}, u_i) \in \Phi$ for every $i \in \{1, \ldots, l\}$.

A *trail* is a walk with no repeated arcs. A node w is *reachable* from v if there exists a trail from v to w of length $l \geq 0$.

A digraph $\mathcal{D} = (V, \Phi)$ is termed *strongly connected* if for every pair of nodes $v, w \in V$, w is reachable from v and v is reachable from w. If $\mathcal{D}$ is not strongly connected, let $U \subset V$: the induced sub-digraph $\mathcal{D}[U]$ is a *strongly connected component* of $\mathcal{D}$ if $\mathcal{D}[U]$ is strongly connected but $\mathcal{D}[U \cup \{v\}]$ is not, for any $v \in V \setminus U$. A strongly connected component $\mathcal{D}[U]$ is *trivial* if it contains a single node without a self-loop, i.e. $\mathcal{D}[U] = (\{u\}, \emptyset)$. Otherwise it is *non-trivial*. The digraph $\mathcal{D}$ is *acyclic* if all its strongly connected component are trivial. We term *acyclic ordering* a relabeling $x_1, x_2, \ldots, x_{|V|}$ of the vertices of $\mathcal{D}$ such that for every arc $(x_i, x_j) \in \Phi$ it holds $j < i$. Any acyclic digraph admits an acyclic ordering [27, Prop 2.1.3].

Given the digraph $\mathcal{D} = (V, \Phi)$ consider all its strongly connected components $\mathcal{D}_k = (V_k, \Phi_k)$, $k \in \{1, \ldots, s\}$. The *condensation digraph* $\mathcal{C}_{\mathcal{D}}$ of $\mathcal{D}$ is the digraph with vertex set $\{1, \ldots, s\}$ where there is an arc from h to k if and only if there is an arc in $\mathcal{D}$ from a node in V_h to a node in V_k and $k \neq h$. It is easy to check that $\mathcal{C}_{\mathcal{D}}$ is acyclic.

We are ready to define the *message digraph* $\mathcal{M}_{\mathcal{G}} = (V, \Phi)$ associated to the graph $\mathcal{G} = (I, E, C)$. The node set of $\mathcal{M}_{\mathcal{G}}$ is $V \subseteq (I \setminus \{\mathfrak{f}\}) \times (I \setminus \{\mathfrak{f}\})$ and contains the *ordered* pairs of vertices of $I \setminus \{\mathfrak{f}\}$ that share an edge in $\mathcal{G}$:

$$V := \{ji : \{i, j\} \in E,\ i \neq \mathfrak{f},\ j \neq \mathfrak{f}\},$$

where $ji := (j, i)$ is a shorthand notation we reserve for the elements of V. The arc set of $\mathcal{M}_{\mathcal{G}}$ is defined by:

$$\Phi := \{(ji, hk) : ji \text{ and } hk \in V,\ i = h,\ j \neq k\},$$

and is inspired by the MPA update rules (5)-(6). Figure 2 illustrates the message digraph $\mathcal{M}_{\mathcal{P}}$ associated to a path $\mathcal{P}$ of four nodes. More in general, the figure shows how a pair of consecutive edges of $\mathcal{G}$ that do not involve the field node $\mathfrak{f}$ map into two arcs of $\mathcal{M}_{\mathcal{G}}$. Note that nodes like ii, self-loops (ji, ji) and arcs like (ji, ij) are never present in the message digraph. We observe without proof that if $\mathcal{G}$ is connected then $\mathcal{M}_{\mathcal{G}}$ enjoys the following properties:

- if $\mathcal{G}$ is an effective tree then $\mathcal{M}_{\mathcal{G}}$ is acyclic;
- if $\mathcal{G}$ contains exactly one cycle that does not include the field node $\mathfrak{f}$ then $\mathcal{M}_{\mathcal{G}}$ contains exactly two non-trivial strongly connected components;
- if $\mathcal{G}$ contains at least two cycles that do not include the field node $\mathfrak{f}$ then $\mathcal{M}_{\mathcal{G}}$ contains exactly one non-trivial strongly connected components.

A complete analysis of the topological properties of $\mathcal{M}_{\mathcal{G}}$ is outside the scope of this paper. We are however interested in the following finer connectivity property, which will be crucial in our argument and which we verify in details.

Lemma 5. *Consider a connected graph $\mathcal{G} = (I, E, C)$, the corresponding massage digraph $\mathcal{M}_{\mathcal{G}} = (V, \Phi)$ and the vector $\boldsymbol{\alpha} \in \mathbb{R}_+^V$ such that $\alpha_{hk} = C_{k\mathfrak{f}}/C_{kh}$ for every $hk \in V$. For every ji in a non-trivial strongly connected component of $\mathcal{M}_{\mathcal{G}}$ there exists hk reachable from ji such that $\alpha_{hk} > 0$.*

Proof. If the node $i \in I$ is a neighbor of $\mathfrak{f}$ in the graph $\mathcal{G}$, the claim is trivially true. In fact, $i \in N_{\mathfrak{f}}$ implies $C_{i\mathfrak{f}} > 0$ while ji in V implies $\{i, j\} \in E$ and $C_{ij} > 0$. Therefore $\alpha_{ji} = C_{i\mathfrak{f}}/C_{ij} > 0$.

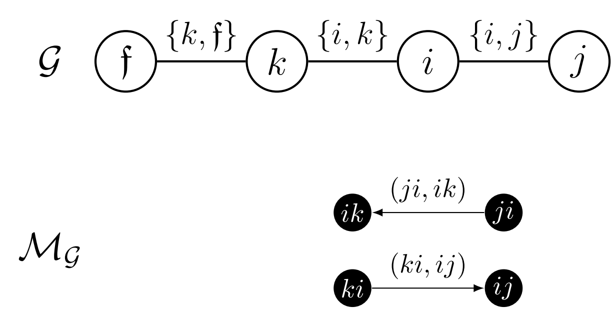

Fig. 2. The path $\mathcal{P} = (\{\mathfrak{f}, k, i, j\}, \{\{\mathfrak{f}, k\}, \{k, i\}, \{i, j\}\})$ (above) and the message digraph $\mathcal{M}_{\mathcal{P}} = (\{ik, ki, ji, ij\}, \{(ji, ik), (ki, ij)\})$ (below). For more general graphs $\mathcal{G}$, to each pair of consecutive edge that do not contain the field node $\mathfrak{f}$ there correspond two arcs in $\mathcal{M}_{\mathcal{G}}$.

If i is not a neighbor of $\mathfrak{f}$ in $\mathcal{G}$, i.e. $i \notin N_{\mathfrak{f}}$, assume that in $\mathcal{M}_{\mathcal{G}}$ the node $ji \in V$ belongs to a non-trivial strongly connected component. The assumption means that there exists in $\mathcal{M}_{\mathcal{G}}$ a trail from ji to itself of length at least 3, because arcs like (ji, ji) and (ji, ij) do not belong to Φ. Correspondingly, $\mathcal{G}$ contains a cycle that includes the edge $\{i, j\}$ and the graph $\mathcal{G} - \{i, j\}$ (i.e. the graph obtained removing the edge $\{i, j\}$ from $\mathcal{G}$) is connected. Hence, $\mathcal{G} - \{i, j\}$ contains a path with endvertices i and $\mathfrak{f}$ of length at least 2: $\{k, \mathfrak{f}\}$ and $\{h, k\}$ are two edges of that path. Such path is also contained in $\mathcal{G}$. Observe that $C_{k\mathfrak{f}} > 0$ and $C_{kh} > 0$ so $\alpha_{hk} > 0$. Therefore, the message digraph $\mathcal{M}_{\mathcal{G}}$ contains a trail $(ji, \ldots, hk)$ from ji to hk and the thesis follows. $\square$

4.3 Convergence of a MPA-like dynamics on digraphs

We define a generalization of the MPA (5)-(6) on directed graphs and prove that it converges on any digraph provided certain conditions are satisfied. The proof is straightforward for acyclic graphs but more involved for graphs that contain strongly connected components. We consider the digraph $\mathcal{D} = (V, \Phi)$ and its adjacency matrix $M \in \{0, 1\}^{V \times V}$, i.e. the matrix such that $M_{vw} = 1$ if and only if $(v, w) \in \Phi$. We consider two positive vectors $\mathbf{r}, \mathbf{s} \in (0, +\infty)^V$ and the matrix $W \in [0, +\infty)^{V \times V}$ defined by:

$$W = \mathrm{Diag}(\mathbf{r}) M \,\mathrm{Diag}(\mathbf{s}).$$

Let the two sequence of non-negative vectors $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t) \in [0, +\infty)^V$ be given. We consider two new vector sequences $\boldsymbol{\omega}(t) \in (0, 1]^V$ and $\boldsymbol{\eta}(t) \in [1, +\infty)^V$ of initial value $\boldsymbol{\omega}(0) = \boldsymbol{\eta}(0) = \mathbf{1}$ and subsequent values defined by the recursions:

$$\omega_v(t+1) = \frac{1}{1 + \alpha_v(t) + \sum_w W_{vw} \left(1 - \omega_w(t)\right)}, \tag{7}$$

$$\eta_v(t+1) = 1 + \beta_v(t) + \textstyle\sum_w M_{vw}\, \omega_w(t)\, \eta_w(t), \tag{8}$$

for every $v \in V$ and $t \geq 0$. We are interested in the convergence properties of $\boldsymbol{\omega}(t)$ and $\boldsymbol{\eta}(t)$.

We make the following assumption, that holds for the rest of this subsection.

Assumption 1. *The vectorial sequence $\boldsymbol{\alpha}(t)$ is non-decreasing in every component and $\boldsymbol{\beta}(t)$ is convergent. The vectors $\mathbf{r}$ and $\mathbf{s}$ satisfy $r_v = s_v^{-1}$ for every $v \in V$.* $\bullet$

In any acyclic digraph $\boldsymbol{\omega}(t)$ and $\boldsymbol{\eta}(t)$ converge since the interdependencies among the components follow an acyclic order and every preceding component converge.

Lemma 6 (Convergence–Acyclic digraphs)**.** *If the digraph $\mathcal{D} = (V, \Phi)$ is acyclic, then the sequence $\boldsymbol{\eta}(t)$ is convergent and the sequence $\boldsymbol{\omega}(t)$ is non-increasing in every component and convergent. Moreover, $\lim_{t\to+\infty} \omega_v(t) < 1$ if and only if there exists w reachable from v such that $\alpha_w(t)$ is non identically zero.*

Proof. Let the subset $S \subseteq V$ contain the sink nodes of the digraph $\mathcal{D}$. Since $\mathcal{D}$ is acyclic S is non-empty [27, Prop 2.1.1]. For $v \in S$ we have $M_{vw} = W_{vw} = 0$ irrespective of w and the update rules (7) and (8) simplify to $\omega_v(t+1) = \frac{1}{1+\alpha_v(t)}$ and $\eta_v(t+1) = 1 + \beta_v(t)$ respectively. Using Assumption 1 the sequence $w_v(t)$ is non-increasing while $\eta_v(t)$ converges. Moreover, $\lim \omega_w(t) < 1$ if and only if $\alpha_v(t)$ is non identically zero.

If there are non-sink nodes, i.e. $V \setminus S$ is non-empty, we introduce an acyclic ordering $x_1, x_2, \ldots, x_{|V|}$ on V such that $\{x_1, \ldots, x_{|S|}\} \equiv S$ and proceed by induction. Let $k \geq 2$ and assume that, for all $i < k$, $\omega_{x_i}(t)$ is non-increasing, $\eta_{x_i}(t)$ converges and moreover $\lim \omega_{x_i}(t) < 1$ if and only if there exists x_j reachable from x_i (where $j \leq i$) such that $\alpha_{x_j}(t)$ is non identically zero. Since $W_{x_k x_i} = 0$ for any $i \geq k$, the update law (7) of $\omega_{x_k}(t)$ is equivalent to:

$$\omega_{x_k}(t+1) = \frac{1}{1 + \alpha_{x_k}(t) + \sum_{i<k} W_{x_k x_i}\,(1 - \omega_{x_i}(t))}\,.$$

The denominator is the sum of non-decreasing terms so $\omega_{x_k}(t)$ is non-increasing, belongs to $(0, 1]$ and converge. Moreover, $\lim \omega_{x_k}(t) < 1$ iff either $\alpha_{x_k}(t)$ is non identically zero or there exists $W_{x_k x_i} > 0$ and $\lim \omega_{x_i}(t) < 1$. Therefore $\lim \omega_{x_k}(t) < 1$ iff there exists x_j reachable from x_k and $\alpha_{x_k}(t)$ is non identically zero. The update law (8) for $\eta_{x_k}(t)$ simplifies to:

$$\eta_{x_k}(t+1) = 1 + \beta_{x_k}(t) + \textstyle\sum_{i<k} M_{x_k x_i}\,\omega_{x_i}(t)\,\eta_{x_i}(t).$$

The sequence $\eta_{x_k}(t)$ converges because its terms are convergent sequences. The thesis follows by induction. □

The absence of cycles is not necessary but has to be compensated by nodes w where $\alpha_w(t)$ is not identically zero. We prove this for strongly connected graphs.

Lemma 7 (Convergence–Strongly connected graphs)**.** *If the digraph $\mathcal{D} = (V, \Phi)$ is strongly connected and there exists v such that $\alpha_v(t)$ is not identically zero the sequences $\boldsymbol{\omega}(t)$ and $\boldsymbol{\eta}(t)$ converge. Moreover, for every $u \in V$ the sequence $\omega_u(t)$ is non-increasing and has limit $\omega_u(\infty) < 1$.*

Proof. We first show that $\boldsymbol{\omega}(t)$ converges and that every component of the limit is strictly smaller than 1. Then, by using the implicit form of the limit, we show that the matrix $M\,\mathrm{Diag}(\boldsymbol{\omega}(t))$ is eventually Shur stable and we conclude that also $\boldsymbol{\eta}(t)$ converges.

Assumption 1 implies that $\boldsymbol{\omega}(t+1) \preccurlyeq \boldsymbol{\omega}(t)$ for every $t \geq 0$. A direct computation gives $\boldsymbol{\omega}(1) \preccurlyeq \boldsymbol{\omega}(0) = \mathbf{1}$ since $\boldsymbol{\alpha}(0) \succcurlyeq \mathbf{0}$. Then, by induction, we let $\boldsymbol{\omega}(t) \preccurlyeq \boldsymbol{\omega}(t-1)$ and deduce that for every $v \in V$:

$$\begin{aligned}\omega_v(t{+}1) &= \frac{1}{1 + \alpha_v(t) + \sum_w W_{vw}\,(1{-}\omega_w(t))} \\ &\leq \frac{1}{1 + \alpha_v(t{-}1) + \sum_w W_{vw}\,(1{-}\omega_w(t{-}1))} = \omega_v(t)\,,\end{aligned}$$

because $\boldsymbol{\alpha}(t) \succcurlyeq \boldsymbol{\alpha}(t-1)$. Consequently, $\boldsymbol{\omega}(t+1) \preccurlyeq \boldsymbol{\omega}(t)$ for every $t \geq 0$ and by monotonicity the sequence admits a limit $\bar{\boldsymbol{\omega}} := \lim_{t\to+\infty} \boldsymbol{\omega}(t)$ that is positive in every component. In order to show that actually $\bar{\omega}_v \in (0, 1)$ for every v, we observe that, by the additional assumption on $\boldsymbol{\alpha}(t)$, there exist $s \geq 0$ and $v \in V$ such that $\boldsymbol{\alpha}(t) = \mathbf{0}$ for $t < s$ whereas $\alpha_v(s) > 0$. Hence, $\boldsymbol{\omega}(t) = \mathbf{1}$ for $t \leq s$ whereas $\boldsymbol{\omega}(s+1) \prec \mathbf{1}$ since $\omega_v(s+1) < 1$. Let us define the set $R_t := \{v : \omega_v(t) < 1\}$, and observe that $R_{s+1} \neq \emptyset = R_s$. If $R_{s+1} = V$, we have shown that $\bar{\omega}_v < 1$ for every v. If $R_{s+1} \neq V$, for $t \geq s+1$ the set R_t is a proper superset of R_{t-1} unless $R_{t-1} \equiv V$. The strong connectivity allows for a pair of nodes v, w such that $v \notin R_{t-1}$, $w \in R_{t-1}$ and $(v, w) \in \Phi$, thus $\omega_v(t) < 1$ and $v \in R_t$. Hence $R_t = V$ eventually.

Next, we prove that $W\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})$ is Schur stable. By hypothesis, the sequence $\boldsymbol{\alpha}(t)$ admits a limit $\bar{\boldsymbol{\alpha}} \succ \mathbf{0}$. The limit $\bar{\boldsymbol{\omega}}$ of the recursion (7) solves, within $(0, 1)^V$, the non-linear system:

$$\bar{\omega}_v = \frac{1}{1 + \bar{\alpha}_v + \sum_w W_{vw}\,(1 - \bar{\omega}_w)} \qquad \forall v \in V\,. \tag{9}$$

Since the denominators are positive, we rewrite (9) as:

$$\bar{\omega}_v\,(1 + \bar{\alpha}_v + \textstyle\sum_w W_{vw}(1 - \bar{\omega}_w)) = 1 \qquad \forall v \in V\,,$$

or equivalently:

$$\textstyle\sum_w \bar{\omega}_v W_{vw}(1 - \bar{\omega}_w) = 1 - \bar{\omega}_v - \bar{\alpha}_v\,\bar{\omega}_v \quad \forall v \in V\,.$$

By the change of variables $x_v := 1 - \bar{\omega}_v$, $c_v := \bar{\alpha}_v\,\bar{\omega}_v$ and $B_{vw} := \bar{\omega}_v\,W_{vw}$ we obtain:

$$\textstyle\sum_w B_{vw}\,x_w = x_v - c_v \qquad \forall v \in V\,,$$

that in vectorial form reads:

$$B\mathbf{x} = \mathbf{x} - \mathbf{c}\,. \tag{10}$$

In the "eigenvalue-like" expression (10), the matrix $B = \mathrm{Diag}(\bar{\boldsymbol{\omega}})W$ is non-negative and irreducible: every component of $\bar{\boldsymbol{\omega}}$ is positive and W is non negative with the positive entries arranged as the adjacency matrix a strongly connected graph, so it is irreducible. Every component of $\mathbf{x}$ is positive and $\mathbf{c} \succ \mathbf{0}$ because every component of $\bar{\boldsymbol{\omega}}$ belongs to $(0, 1)$ and $\bar{\alpha} \succ \mathbf{0}$. If we multiply (10) on the left by $B^{|V|-1}$ and iteratively reuse (10), we obtain:

$$B^{|V|}\mathbf{x} = \mathbf{x} - \textstyle\sum_{i=0}^{|V|-1} B^i\,\mathbf{c}\,.$$

Every element of the matrix $\sum_{i=0}^{|V|-1} B^i$ is positive, because B is non-negative and irreducible [28, Corollary on p. 52]. Therefore, every component of $\sum_{i=0}^{|V|-1} B^i\,\mathbf{c}$ is positive and:

$$(B^{|V|}\mathbf{x})_v < x_v \qquad \forall v \in V\,,$$

which implies that the spectral radius of $B^{|V|}$ is strictly smaller than one [29, Lemma 34.7], i.e. $\rho(B^{|V|}) < 1$. Thus, $\rho(B) < 1$ and since $B = \mathrm{Diag}(\bar{\boldsymbol{\omega}})W$ and $W\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})$ have the same eigenvalues:

$$\rho\,(W\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})) < 1\,. \tag{11}$$

We finally show that $\boldsymbol{\eta}(t)$ converges. Assumption 1 (i.e. $\mathrm{Diag}(\mathbf{r}) = \mathrm{Diag}(\mathbf{s})^{-1}$) implies that the matrix $W\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})$ and $M\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})$ are similar:

$$\begin{aligned}W\,\mathrm{Diag}(\bar{\boldsymbol{\omega}}) &= \mathrm{Diag}(\mathbf{r})M\,\mathrm{Diag}(\mathbf{s})\,\mathrm{Diag}(\bar{\boldsymbol{\omega}}) \\ &= \mathrm{Diag}(\mathbf{s})^{-1}M\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})\,\mathrm{Diag}(\mathbf{s})\end{aligned}$$

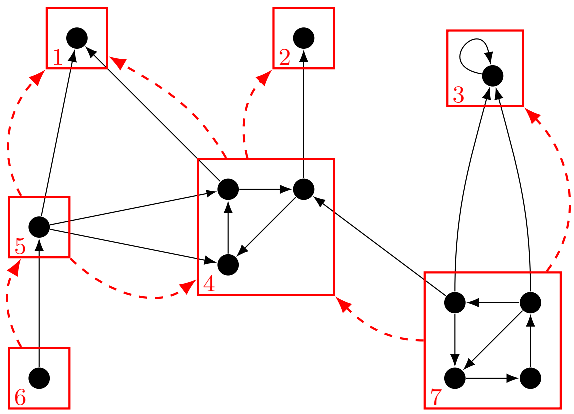

Fig. 3. A digraph $\mathcal{D} = (V, \Phi)$ and its condensation digraph $\mathcal{C}_{\mathcal{D}}$. The digraph $\mathcal{D}$ has black round nodes and thin arcs. The condensation digraph $\mathcal{C}_{\mathcal{D}}$ has box nodes and dashed edges. The numbers in the nodes of $\mathcal{C}_{\mathcal{D}}$ form an acyclic order.

and thus:

$$\rho\,(M\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})) < 1\,.$$

The matrix $M\,\mathrm{Diag}(\boldsymbol{\omega}(t))$ converges to $M\,\mathrm{Diag}(\bar{\boldsymbol{\omega}})$ hence it is eventually Schur stable. In vectorial form, the update law (8) reads:

$$\boldsymbol{\eta}(t+1) = \mathbf{1} + \boldsymbol{\beta}(t) + M\ \mathrm{Diag}(\boldsymbol{\omega}(t))\,\boldsymbol{\eta}(t)\,,$$

where the sequence $\boldsymbol{\beta}(t)$ converges and so does $\boldsymbol{\eta}(t)$. $\square$

For strongly connected digraphs the presence of at least one node v with $\alpha_v(t)$ non identically zero is necessary to make the sequence $\boldsymbol{\eta}(t)$ converge. If such a node is not present, then $\boldsymbol{\omega}(t) = \mathbf{1}$ for every $t \geq 0$ and since M is irreducible $\rho(M\,\mathrm{Diag}(\boldsymbol{\omega}(t))) = \rho(M) \geq 1$ so $\boldsymbol{\eta}(t)$ grows unbounded.

More in general, the vector sequences $\boldsymbol{\omega}(t)$ and $\boldsymbol{\eta}(t)$ converge on any digraph $\mathcal{D}$ provided that for any node in a strongly connected component there exists a reachable node w such that $\alpha_w(t)$ is non identically zero. To prove the statement we consider the condensation graph $\mathcal{C}_{\mathcal{D}}$ of the digraph $\mathcal{D}$ and fix an acyclic order on it, see Figure 3. Within any strongly connected component (trivial or not) the dynamic converges following the acyclic order; the convergence of the remaining components follows. The sequences $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ introduced before the definition of the recursive laws (7) and (8) are used here to "connect" the different components of the digraph.

Proposition 8 (Convergence–General graphs)**.** *Consider any digraph $\mathcal{D} = (V, \Phi)$ and the vector sequences $\boldsymbol{\omega}(t)$ and $\boldsymbol{\eta}(t)$ defined with the recursive laws* (7)-(8)*. Assume that, for every node v that belongs to a non-trivial strongly connected component of $\mathcal{D}$, there exists a node w reachable from v such that the sequence $\alpha_w(t)$ is non identically zero. Then, the sequence $\boldsymbol{\eta}(t)$ converges and the sequence $\boldsymbol{\omega}(t)$ converges and is non-increasing in every component. Moreover $\lim_{t\to+\infty} \omega_v(t) < 1$ for every node v such that there exists w reachable from v and $\alpha_w(t)$ is not identically zero.*

Proof. Consider the condensation graph $\mathcal{C}_{\mathcal{D}}$ of $\mathcal{D}$. Let $\{1, 2\ldots, s\}$ be the vertex set of $\mathcal{C}_{\mathcal{D}}$ and assign the nodes' label to form an acyclic order on $\mathcal{C}_{\mathcal{D}}$ where the smallest number are reserved to sink nodes, c.f. Figure 3. Assume k is the node of $\mathcal{C}_{\mathcal{D}}$ that represents the strongly connected component $\mathcal{D}_k = (V_k, \Phi_k)$. For every $v \in V_k$ and $t \geq 0$, we rewrite the recursive laws (7)-(8) as:

$$\omega_v(t+1) = \frac{1}{1 + \alpha'_v(t) + \sum_{w\in V_k} W_{vw}\,(1-\omega_w(t))}\,, \tag{12}$$

$$\eta_v(t+1) = 1 + \beta'_v(t) + \textstyle\sum_{w\in V_k} M_{vw}\,\omega_w(t)\,\eta_w(t)\,, \tag{13}$$

where:

$$\alpha'_v(t) := \alpha_v(t) + \textstyle\sum_{w\notin V_k} W_{vw}\,(1-\omega_w(t))\,, \tag{14}$$

$$\beta'_v(t) := \beta_v(t) + \textstyle\sum_{w\notin V_k} M_{vw}\,\omega_w(t)\,\eta_w(t)\,. \tag{15}$$

Let k be a sink node of $\mathcal{C}_{\mathcal{D}}$ (there must be at least one) and observe that $M_{v,w} = W_{v,w} = 0$ for any $v \in V_k$ and $w \notin V_k$. Hence $\alpha'_v(t) = \alpha_v(t)$ and $\eta'_v(t) = \eta_v(t)$ for any $v \in V_k$ and $t \geq 0$ so the dynamics within the component $\mathcal{D}_k$ is independent of any other component. Therefore the sequences $\omega_v(t)$ and $\eta_v(t)$ converge for any $v \in V_k$: if $\mathcal{D}_k$ is a non-trivial strongly connected component, invoke Lemma 7; else $\mathcal{D}_k = (\{v\}, \emptyset)$ and it is sufficient to observe the expressions, similar to those in the proof of Lemma 6. Moreover, $\omega_v(t)$ is non-increasing and $\lim \omega_v(t) < 1$ if there is $w \in V_k$ such that $\alpha_w(t)$ is non identically zero.

Consider now any non-sink node $k > 1$ of $\mathcal{C}_{\mathcal{D}}$ and assume that the sequences $\omega_u(t)$ and $\eta_u(t)$ converge for any node $u \in V_h$ in any component $\mathcal{D}_h$ where $h < k$. Assume moreover that $\lim \omega_u(t) < 1$ if there exists w reachable from u such that $\alpha_w(t)$ is non identically zero. Let $v \in V_k$ and observe that the sequence $\alpha'_v(t)$ and $\eta'_v(t)$ defined in (14)-(15) only contain terms $\omega_u(t)$ and $\eta_u(t)$ where $u \in V_h$ for some $h < k$. Given these assumptions $\alpha'_v(t)$ is non-decreasing and, if there exists in $\mathcal{D}$ a node w reachable from v such that $\alpha_w(t)$ is non identically zero, non identically zero. Moreover, $\beta'_v(t)$ converges.

Therefore, by inspection if $\mathcal{D}_k$ is trivial or using Lemma 7 if $\mathcal{D}_k$ is non-trivial, the sequences $\omega_v(t)$ and $\eta_v(t)$ converge for any $v \in V_k$, $\omega_v(t)$ is non-increasing and, if there exists in $\mathcal{D}$ a node w reachable from v such that $\alpha_w(t)$ is non identically zero, $\lim \omega_v(t) < 1$. An induction on the remaining components of $\mathcal{C}_{\mathcal{D}}$ proves the claim. $\square$

4.4 Convergence of the MPA on $\mathcal{G}$

We are now ready to prove Theorem 4 by applying Proposition 8: this requires to verify that Assumption 1 is satisfied. The argument below hinges on the connectivity properties of $\mathcal{M}_{\mathcal{G}}$ established in Lemma 5.

Proof of Theorem 4. We simplify the recursive laws (5)-(6) of the MPA on $\mathcal{G} = (I, E, C)$ by excluding the messages sent or received by the field node $\mathfrak{f}$. In fact, the messages sent by the field node $\mathfrak{f}$ are zero constants:

$$W^{\mathfrak{f}\to j}(t) = H^{\mathfrak{f}\to j}(t) = 0\,, \quad \forall j \in N_{\mathfrak{f}}, \quad \forall t \geq 0\,,$$

and substituting them in the recursive laws we obtain:

$$W^{i\to j}(t+1) = \left(1+\frac{C_{i\mathfrak{f}}}{C_{ij}}+\sum_{k\in N_i\setminus\{j,\mathfrak{f}\}}\frac{C_{ik}}{C_{ij}}\Big(1-W^{k\to i}(t)\Big)\right)^{-1} \tag{16}$$

$$H^{i\to j}(t+1) = 1+\sum_{k\in N_i\setminus\{j,\mathfrak{f}\}} W^{k\to i}(t)\,H^{k\to i}(t)\,. \tag{17}$$

The messages sent to the field node $\mathfrak{f}$ play no role because no message depends on $W^{i\to\mathfrak{f}}(t)$ and $H^{i\to\mathfrak{f}}(t)$. Hence the messages $W^{i\to j}(t)$ and $H^{i\to j}(t)$ with $i,j\neq\mathfrak{f}$ form an autonomous system.

Consider now the message digraph $\mathcal{M}_{\mathcal{G}}=(V,\Phi)$ associated to the graph $\mathcal{G}=(I,E,C)$ and on it the dynamics of the vector sequences $\boldsymbol{\omega}(t)$ and $\boldsymbol{\eta}(t)$ described at the beginning of Section 4.3. Let M be the adjacency matrix of $\mathcal{M}_{\mathcal{G}}$ and the vectors $\mathbf{r}$ and $\mathbf{s}$ be such that:

$$r_{ji}=(C_{ij})^{-1}\,,\quad s_{ji}=C_{ji}\,,\quad \forall ji\in V\,.$$

The vector sequences $\boldsymbol{\omega}(t)$ and $\boldsymbol{\eta}(t)$ have initial value $\boldsymbol{\omega}(0)=\boldsymbol{\eta}(0)=\mathbf{1}$ and subsequent values given by the following recursive laws, valid for every $ji\in V$ and $t\geq 0$:

$$\omega_{ji}(t+1)=\frac{1}{1+\alpha_{ji}(t)+\sum_{hk\in V}W_{ji,hk}\,(1-\omega_{hk}(t))}\,, \tag{18}$$

$$\eta_{ji}(t+1)=1+\beta_{ji}(t)+\textstyle\sum_{hk\in V}M_{ji,hk}\,\omega_{hk}(t)\,\eta_{hk}(t)\,, \tag{19}$$

where $W_{ji,hk}=r_{ji}\,M_{ji,hk}\,s_{hk}$ and for every $t\geq 0$ the vector sequence $\boldsymbol{\beta}(t)=\mathbf{0}$ while $\boldsymbol{\alpha}(t)$ satisfies:

$$\alpha_{ji}(t)=C_{i\mathfrak{f}}/C_{ij}\,,\quad \forall ji\in V\,.$$

Comparing $\omega_{ji}(t)$ and $\eta_{ji}(t)$ and their laws (18)-(19) with $W^{i\to j}(t)$ and $H^{i\to j}(t)$ and their laws (16)-(17) we recognize that:

$$W^{i\to j}(t)\equiv\omega_{ji}(t)\qquad H^{i\to j}(t)\equiv\eta_{ji}(t)\,,$$

for every $ji\in V$ and $t\geq 0$. In other words, the message $W^{i\to j}(t)$ that i sends to j (with $i,j\neq\mathfrak{f}$) corresponds to the sequence $\omega_{ji}(t)$ and similarly $H^{i\to j}(t)$ corresponds to $\eta_{ji}(t)$, see the example in Figure 4. According to the MPA's update rules (18)-(19) they depend on the messages $W^{k\to i}(t)$ and $H^{k\to i}(t)$ where $k\in N_i\setminus\{j,\mathfrak{f}\}$: the arc $(ji,ik)\in\Phi$ represents such dependence relation.

The vectorial sequences $\boldsymbol{\alpha}(t),\boldsymbol{\beta}(t)$ and the vectors $\mathbf{r},\mathbf{s}$ satisfy Assumption 1 because $\boldsymbol{\alpha}(t),\boldsymbol{\beta}(t)$ are constant while:

$$r_{ji}=(C_{ij})^{-1}=(C_{ji})^{-1}=s_{ji}^{-1}\quad\text{for every } ji\in V\,,$$

since the matrix C is symmetric. Using Lemma 5 the connectivity of $\mathcal{G}$ implies that the hypothesis of Proposition 8 are satisfied and the dynamic on $\mathcal{M}_{\mathcal{G}}$ converge. Then, every message of the MPA on $\mathcal{G}$ converge (also the messages received by $\mathfrak{f}$) and we conclude that the sequence $H^{\ell}(t)$ converges for every node $\ell\in I\setminus\{\mathfrak{f}\}$. □

Finally we observe that the symmetry of the matrix C is *not necessary* to prove the convergence of the messages $W^{i\to j}(t)$ on connected graphs $\mathcal{G}=(I,E,C)$. We will discuss the convergence of the corresponding messages $H^{i\to j}(t)$ in the next section using numerical simulations.

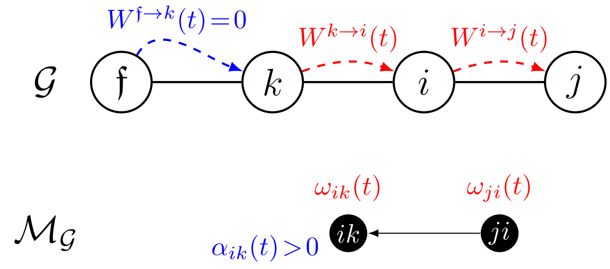

Fig. 4. The messages $W^{k\to i}(t)$ and $W^{i\to j}(t)$ drawn in red in the path $\mathcal{P}$ (above) have as corresponding counterparts in $\mathcal{M}_{\mathcal{P}}$ (below) the black nodes ik and ji and the red sequences $\omega_{ik}(t)$ and $\omega_{ji}(t)$. Not represented, also the the messages $W^{i\to k}(t)$ and $W^{j\to i}(t)$ have corresponding counterparts in $\mathcal{M}_{\mathcal{P}}$. The counterpart of the message $W^{\mathfrak{f}\to k}(t)=0$ drawn in blue is the blue (constant) sequence $\alpha_{ik}(t)>0$. The message $W^{k\to\mathfrak{f}}(t)$ has no counterpart and is not drawn. The picture is similar for the messages $H^{i\to j}(t)$.

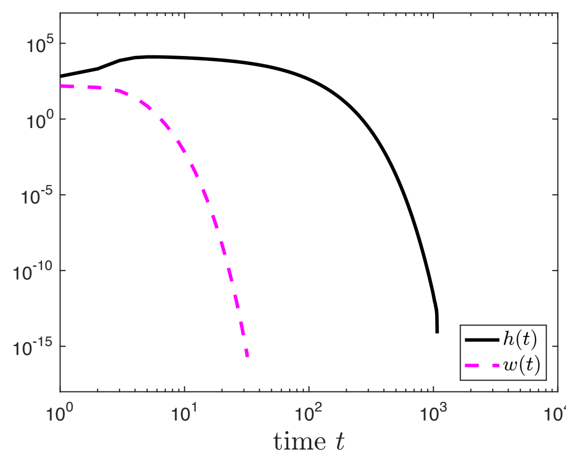

Fig. 5. The MPA convergence on graph $\mathcal{G}_1$. The solid black line is $h(t)$, the 1-norm distance between the estimates of the harmonic influence $H^{\ell}(t)$ at time t and their corresponding limits $H^{\ell}(\infty)$. The dashed magenta line is $w(t)$, i.e. the 1-norm distance between $W^{i\to j}(t)$ with $i,j\neq\mathfrak{f}$ and their corresponding limits $W^{i\to j}(\infty)$.

Proposition 9 (Convergence without symmetry)**.** *Consider the connected graph $\mathcal{G}=(I,E,C)$ and let $\mathcal{M}_{\mathcal{G}}=(V,\Phi)$ be the corresponding message digraph. Then, the messages $W^{i\to j}(t)$ converge and, moreover,*

$$\rho\big(\operatorname{Diag}(\mathbf{r})M\operatorname{Diag}(\bar{\boldsymbol{\omega}})\operatorname{Diag}(\mathbf{s})\big)<1\,,$$

where M is the adjacency matrix of $\mathcal{M}_{\mathcal{G}}$ and the components of vectors $\mathbf{r},\mathbf{s},\bar{\boldsymbol{\omega}}$ are

$$r_{ji}=(C_{ij})^{-1}\,,\quad s_{ji}=C_{ji}\,,\quad \bar{\omega}_{ji}=W^{i\to j}(\infty)\,.$$

Proof. The proof follows the same line of the proof of Theorem 4. In general the matrix C is not symmetric and the vectors $\mathbf{r}$ and $\mathbf{s}$ do not satisfy $r_v=s_v^{-1}$ for every $v\in V$ so the second part of Assumption 1 is not verified and Proposition 8 cannot be used as is. However note that the convergence of the vector sequence $\boldsymbol{\omega}(t)$ on the message digraph does not depend on that part of the Assumption and so the convergence of the messages $W^{i\to j}(t)$.

To prove the second part of the claim consider the condensation graph $\mathcal{C}_{\mathcal{M}_{\mathcal{G}}}$ of the message digraph $\mathcal{M}_{\mathcal{G}}$, group and reorder the rows and column of M according to the partial order that $\mathcal{C}_{\mathcal{M}_{\mathcal{G}}}$ implies between the strongly connected components V_k of $\mathcal{M}_\mathcal{G}$. Similarly do for the vectors $\mathbf{r}, \mathbf{s}$ and $\bar{\boldsymbol{\omega}}$. The matrix $W \operatorname{Diag}(\bar{\boldsymbol{\omega}})$, where $W = \operatorname{Diag}(\mathbf{r}) M \operatorname{Diag}(\mathbf{s})$, is a block lower triangular matrix. Trivial strongly connected components contain a single node and the corresponding block is simply 0. For non trivial strongly connected components V_k the equation (11) in the proof of Lemma 7 holds, giving

$$\rho\big((W \operatorname{Diag}(\bar{\boldsymbol{\omega}}))_{V_k,V_k}\big) < 1\,,$$

because it does not require the second part of Assumption 1. The claim follows from the block lower triangular matrix structure. □

The argument in this proof fails to guarantee convergence of $H^{i\to j}$ because without the second part of Assumption 1 matrices $M \operatorname{Diag}\bar{\boldsymbol{\omega}}$ and $W \operatorname{Diag}\bar{\boldsymbol{\omega}}$ are not similar. However, numerical experiments (presented in the next section) suggest that, irrespective of the symmetry of C, $\rho\big(M \operatorname{Diag}(\bar{\boldsymbol{\omega}})\big) = \rho\big(W \operatorname{Diag}(\bar{\boldsymbol{\omega}})\big)$ and the MPA converges.

5 Simulations on random graphs

In this paper we consider simple weighted graphs $\mathcal{G} = (I, E, C)$ with node set $I = \{\mathfrak{f}, 1, 2, \ldots, n\}$ of cardinality $n+1$. The non-negative weight matrix C is such that $C_{ij} = 0$ if and only if $\{i, j\} \notin E$, thus the main diagonal is null and the zeros are symmetrically located. We know from Proposition 3 that the MPA converges in a finite time to the exact harmonic influence values if $\mathcal{G}$ is an effective tree, and from Theorem 4 that the MPA converges if $\mathcal{G}$ is connected and C symmetric. In this section we first present some numerical simulations which suggest that the symmetricity of the matrix C is not necessary for convergence, see Section 5.1. Then we investigate how convergence time and approximation error depend on the amount of cycles present in the graph, focusing on graphs $\mathcal{G}$ where C is symmetric and the $\mathfrak{f}$ node is connected to every other node, see Section 5.2.

We start by introducing some useful notation. Provided we approximate the asymptotic values $H^\ell(\infty)$ and $W^{i\to j}(\infty)$ by the values of $H^\ell(t)$ and $W^{i\to j}(t)$ after a sufficiently large number of iterations, we can introduce 1-norm distances to the asymptotic values that we will use to check the speed of convergence of the MPA:

$$h(t) = \sum_{\ell=1}^{n} \left| H^\ell(t) - H^\ell(\infty) \right|\,, \tag{20}$$

$$w(t) = \sum_{i\neq\mathfrak{f}} \sum_{\substack{j\neq\mathfrak{f}\\ \{i,j\}\in E}} \left| W^{i\to j}(t) - W^{i\to j}(\infty) \right|\,. \tag{21}$$

In order to assess the approximation of the harmonic influence achieved by the MPA, we plot $H^\ell(\infty)$ against their corresponding exact values $H(\ell)$ computed using definition (2) and a standard solver. Spearman's rank-order correlation coefficient [30] between the two variables is used to give a quantitative evaluation of how much the rankings are preserved. Similarly, we plot $W^{i\to j}(\infty)$ against the value of x_i in the solution of the Laplacian system 1 where $j = \ell$, denoted by $x_i|_{j=\ell}$. Indeed, recall that on effective trees $W^{i\to j}(\infty) = x_i|_{j=\ell}$.

5.1 Convergence for non-symmetric matrices C

We present a group of simulations to show that the MPA converges on general connected graphs $\mathcal{G} = (I, E, C)$. The node set is $I = \{\mathfrak{f}, 1, 2, \ldots, n\}$ with $n = 50$. The edge set is generated randomly: each edge $\{i, j\}$ is present with probability $p = 0.100$ and disconnected graphs are discarded. The entries of the matrix C are chosen as:

$$\begin{cases} C_{ij} = U_{[2,8]} & \text{if } \{i,j\} \in E \\ C_{ij} = 0 & \text{if } \{i,j\} \notin E \end{cases}$$

where $U_{[2,8]}$ is a uniform random variable with support $[2, 8]$. We have observed that all these simulations converge.

We then describe one of these simulation. The generated graph $\mathcal{G}_1$ has 117 edges, making the average degree be 4.6 while the diameter is 5. The degree of the field node is 5 and coincides with the expected degree pn. The non-zero values of C belong to $[2.030, 9.983]$. Figure 5 shows the convergence of the MPA. The distance $w(t)$ between the $W^{i\to j}(t)$ messages and their final values becomes negligible after 30 iterations. The distance $h(t)$ between $H^\ell(t)$ and the final approximation of the harmonic influence requires about 1000 iterations to become negligible. If we rewrite the MPA using the corresponding message digraph we observe that the spectral radius of the matrices $M \operatorname{Diag}(\boldsymbol{\omega}(\infty))$ coincides with that of $W \operatorname{Diag}(\boldsymbol{\omega}(\infty))$ and is strictly smaller than one:

$$\rho\big(M \operatorname{Diag}(\boldsymbol{\omega}(\infty))\big) = \rho\big(W \operatorname{Diag}(\boldsymbol{\omega}(\infty))\big) = 0.964\,.$$

In order to assess the approximation of the harmonic influence achieved by the MPA, in Figure 6 we plot $H^\ell(\infty)$ against their corresponding exact values $H(\ell)$. If the MPA algorithm would be exact, the two vectors would coincide and the pairs $(H(\ell), H^\ell(\infty))$ would be plotted on the 45° line of the diagram. Due to the presence of cycles, the MPA is not exact and overestimates the harmonic influence, see Figure 6 where all crosses are above the 45° line, a behaviour consistently observed throughout simulations. However, the points $(H(\ell), H^\ell(\infty))$ approximately form a monotonically increasing function, meaning that the nodes' rankings are fairly preserved: indeed, the Spearman correlation coefficient is 0.977. Similarly, the crosses in Figure 7 represent the values $W^{i\to j}(\infty)$ plotted against the exact values of their interpretation $x_i|_{j=\ell}$. The points form an elongated cloud and are below the 45° line: the limit values $W^{i\to j}(\infty)$ are consistently smaller than the corresponding $x_i|_{j=\ell}$.

Even though the approximation provided by the algorithm is usually fairly good, there are extreme cases where either the algorithm fails to provide a good answer or, on the contrary, is particularly effective. We provide two corresponding examples next.

If field node $\mathfrak{f}$ is a leaf of the connected random graph, i.e. it has a unique neighbor k and $N_\mathfrak{f} = \{k\}$, it is easy to see from the definition that $H(k) = n$, the highest possible value. The corresponding solution of the Laplacian system (1) where $\ell = k$ is in fact the all-one vector except $x_\mathfrak{f} = 0$, irrespective of the weights in the matrix C. We then run our algorithm on one such graph (which we call $\mathcal{G}_2$, with $C_{k\mathfrak{f}} = 4.291$). The convergence of $h(t)$ is very slow and takes around 10000 steps, because $\rho\big(W \operatorname{Diag}(\boldsymbol{\omega}(\infty))\big) = \rho\big(M \operatorname{Diag}(\boldsymbol{\omega}(\infty))\big) = 0.996$. Figure 8 shows the harmonic influences computed by the MPA against their exact values

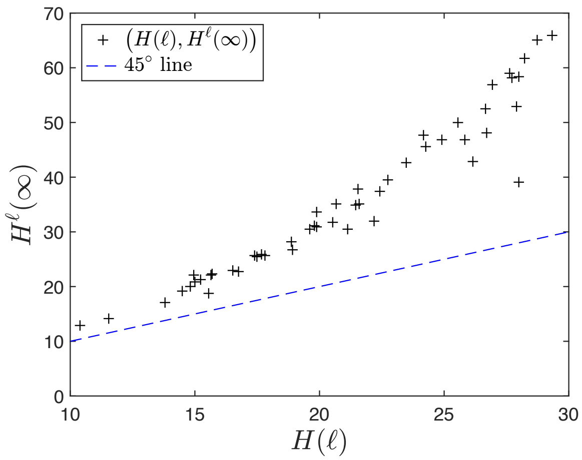

Fig. 6. The asymptotic values $H^\ell(\infty)$ of the harmonic influence computed by the MPA against the corresponding exact values $H(\ell)$ for the graph $\mathcal{G}_1$. All crosses are above the $45°$ line.

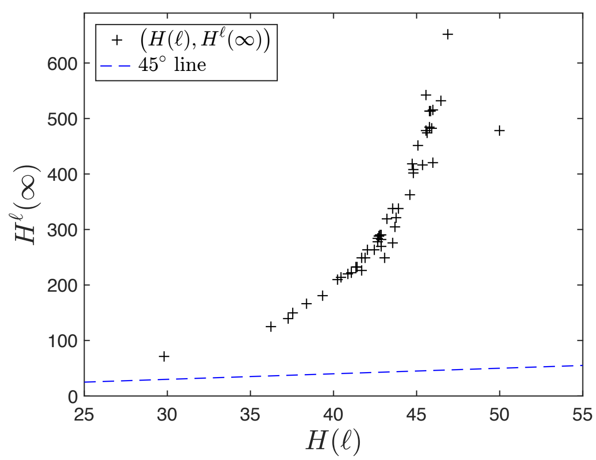

Fig. 8. The asymptotic values $H^\ell(\infty)$ of the harmonic influence computed by the MPA against the corresponding exact values $H(\ell)$ computed by the definition, for the graph $\mathcal{G}_2$. All crosses are above the $45°$ line; the cross in $(50, 479)$ stands out of the cloud.

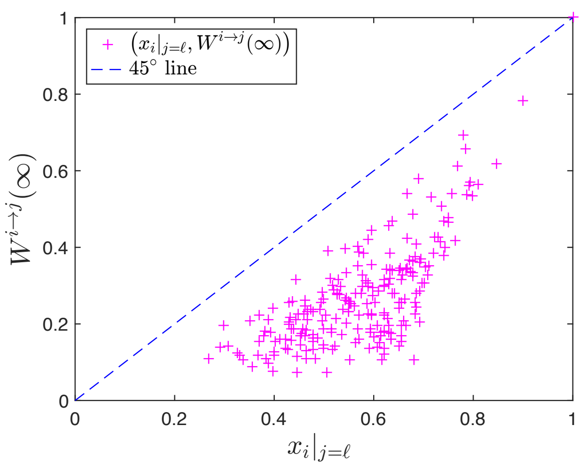

Fig. 7. The asymptotic values of the messages $W^{i\to j}(\infty)$ for $i, j \neq \mathfrak{f}$ against the values $x_i|_{j=\ell}$, i.e. the values of the i^{th} element of the solution of the Laplacian system (1) where $\ell \equiv j$, for the graph $\mathcal{G}_1$. All magenta crosses are below or on the $45°$ line.

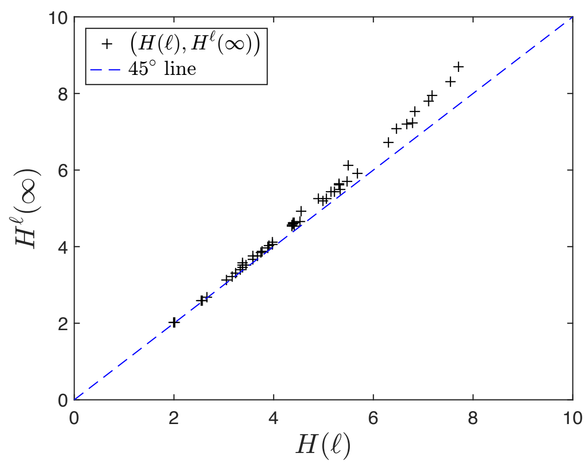

Fig. 9. The asymptotic values $H^\ell(\infty)$ of the harmonic influence computed by the MPA against the corresponding exact values $H(\ell)$ computed by the definition, for the graph $\mathcal{G}_3$. All crosses are close to (but above of) the $45°$ line.

from the definition. The cross $(H(k), H^k(\infty)) = (50, 479)$ stands out of the cloud while all the other crosses are fairly monotonically aligned: the Spearman correlation is 0.972. The MPA misses the fact that the node k has, for topological reasons, the highest harmonic influence.

In the second special case the field node is connected to every other node so $|N_\mathfrak{f}| = n$ (we call $\mathcal{G}_3$ the graph of this simulation). In this case, the convergence is much faster and the approximation is very good. The distance $h(t)$ takes 150 steps to converge while $w(t)$ takes 20 and $\rho\big(M\,\mathrm{Diag}(\boldsymbol{\omega}(\infty)\big) = \rho\big(W\,\mathrm{Diag}(\boldsymbol{\omega}(\infty))\big) = 0.760$. The crosses $(H(\ell), H^k(\ell))$ in Figure 9 are above but very close to the $45°$ line meaning that the harmonic influences computed by the MPA is close to the exact values and almost monotonically aligned: the Spearman correlation is 0.998.

5.2 Cycles in $\mathcal{G}$ and performance of the MPA

In this section, we investigate the effect of the number of cycles on the convergence time and error of the MPA. Since Proposition 3 guarantees finite-time convergence and correctness of the algorithm, we expect that more cycles should result in worse algorithm performance, meaning both slower convergence and larger error. This intuition is confirmed by the following simulations, which are obtained on connected graphs $\mathcal{G} = (I, E, C)$ where the field node is connected to all other nodes, i.e. $\{i, \mathfrak{f}\} \in E$ for every i, and matrix C is symmetric, so that convergence is guaranteed by Theorem 4.

We extract at random the graph $\mathcal{G}_4$ as follows. The node set is $I = \{\mathfrak{f}, 1, 2, \ldots, n\}$ with $n = 50$; the edges $\{i, j\}$ with $i, j \neq \mathfrak{f}$ have a probability $p = 0.100$ of being present and we make sure $\mathcal{G}_4[\{1, \ldots, n\}]$ is connected. The entries of C are:

$$\begin{cases} C_{i\mathfrak{f}} = C_{\mathfrak{f}i} = 0.040 & \text{for every } i \in \{1, \ldots, n\} \\ C_{ij} = 1 & \text{if } i, j \neq \mathfrak{f} \text{ and } \{i, j\} \in E \\ C_{ij} = 0 & \text{if } \{i, j\} \notin E \end{cases} \tag{22}$$

The graph $\mathcal{G}_4$ that we select for the simulation contains 173 edges and has diameter 2. The subgraph $\mathcal{G}_4[\{1, \ldots, n\}]$ is

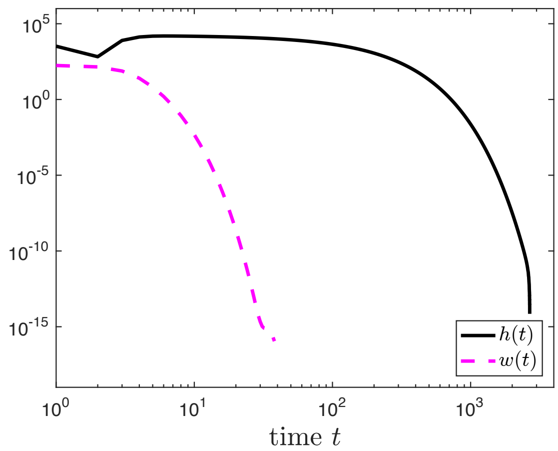

Fig. 10. The MPA convergence on graph $\mathcal{G}_4$. The solid black line is $h(t)$, i.e. the distance to convergence of the estimates of the harmonic influence obtained by the MPA. The dashed magenta line is $w(t)$, i.e. the distance to convergence of the messages $W^{i\to j}(t)$.

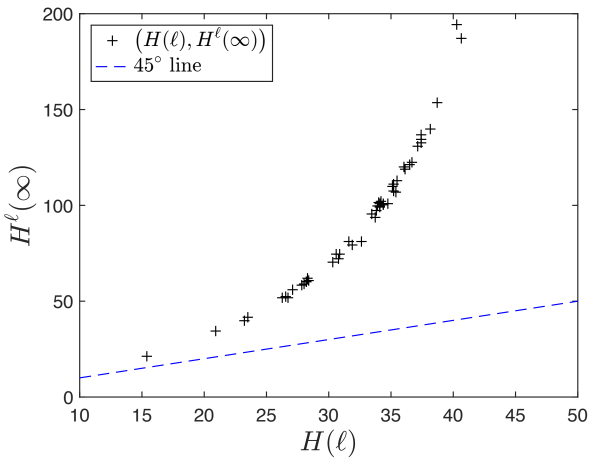

Fig. 11. The asymptotic values $H^\ell(\infty)$ of the harmonic influence computed by the MPA against the corresponding exact values $H(\ell)$ computed by the definition, for the graph $\mathcal{G}_4$. All crosses are above the $45°$ line.

actually a connected realization of a *Erdős-Rényi* random graph with diameter 5 and 123 edges, forming many cycles.

Figure 10 shows the convergence time of the MPA: the distance $w(t)$ becomes negligible after 30 iterations while the distance $h(t)$ requires about 2500 iterations to become negligible. The MPA is not exact: Figure 11 represents $H^\ell(\infty)$ against the corresponding $H(\ell)$, showing that the largest value of $H^\ell(\infty)$ is about 5 times bigger than the corresponding $H(\ell)$. All crosses are nearly aligned above the $45°$ line and Spearman's coefficient is 0.9939: we can say that the nodes' rankings are nearly preserved. The crosses in Figure 12 represent $W^{i\to j}(\infty)$ against the values of $x_i|_{j=\ell}$: all of them are below the $45°$ line.

We repeat the simulations on graph $\mathcal{G}_5$ obtained from $\mathcal{G}_4$ by removing some edge $\{i,j\}$ where $i,j \neq \mathfrak{f}$ so that the subgraph $\mathcal{G}_5[\{1,\dots,n\}]$ is still connected but has fewer cycles than the subgraph $\mathcal{G}_4[\{1,\dots,n\}]$ Matrix C of $\mathcal{G}_5$ is adapted accordingly. The subgraph $\mathcal{G}_5[\{1,\dots,n\}]$ of the simulation has 59 edges for 50 nodes so it contains 10 edges more than a tree which form a few cycles, and has diameter

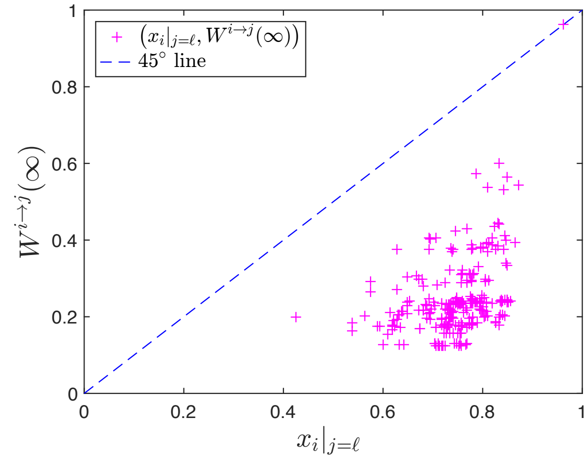

Fig. 12. The asymptotic values of the messages $W^{i\to j}(\infty)$ for $i,j \neq \mathfrak{f}$ against the values $x_i|_{j=\ell}$, for the graph $\mathcal{G}_4$. All magenta crosses are below or on the $45°$ line.

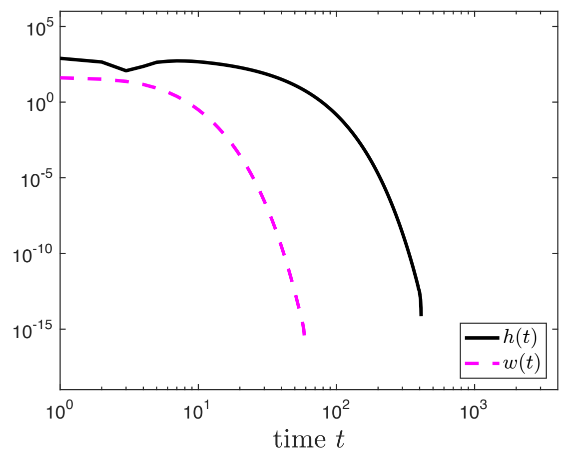

Fig. 13. The MPA convergence on graph $\mathcal{G}_5$. The solid black line is $h(t)$; the dashed magenta line is $w(t)$.

9. Figure 13 shows the convergence time of the MPA. The distance $w(t)$ becomes negligible after 60 iterations, whereas $h(t)$ after about 400 iterations, much less than the previous simulation. Also on this graph the MPA is not exact but the nodes' rankings implied by the harmonic influence are nearly preserved. Figure 14 represents $H^\ell(\infty)$ against the corresponding exact values $H(\ell)$. All crosses are above the $45°$ line and the Spearman's coefficient is 0.9940. The crosses in Figure 15 compare $W^{i\to j}(\infty)$ against the corresponding values $x_i|_{j=\ell}$: all points are just below or on the $45°$ line.

5.3 Size of $\mathcal{G}$ and performance of the MPA

An important motivation behind the development of the MPA is scalability. In this section we define the convergence time of the MPA and simulate it on two families of graphs that generalize those used in Section 5.2 to different sizes.

We define the convergence time of both the estimates $H^\ell(t)$ and the messages $W^{i\to j}(t)$, using the 1-norm distances $h(t)$ and $w(t)$ introduced in (20)-(21)

$$t_{h,\epsilon} := \inf\left\{t : \frac{h(t)}{n} \leq \epsilon\right\}, \quad t_{w,\epsilon} := \inf\left\{t : \frac{w(t)}{2m} \leq \epsilon\right\},$$

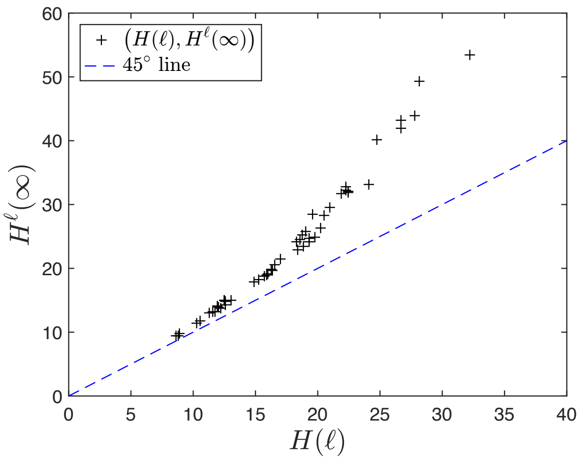

Fig. 14. The asymptotic values $H^\ell(\infty)$ of the harmonic influence computed by the MPA against the corresponding exact values $H(\ell)$ computed by the definition, for graph $\mathcal{G}_5$. All crosses are above the 45° line.

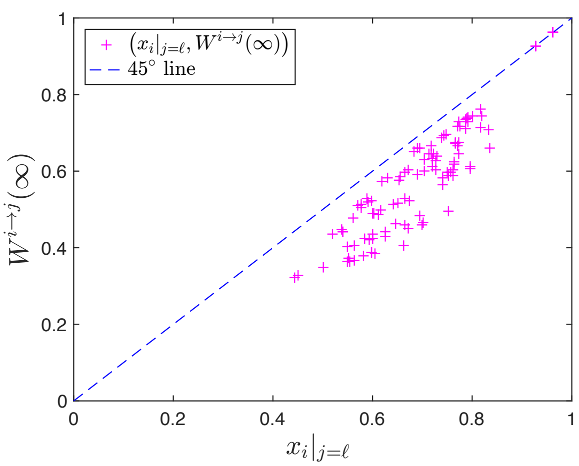

Fig. 15. The asymptotic values of the messages $W^{i\to j}(\infty)$ for $i, j \neq \mathfrak{f}$ against the values $x_i|_{j=\ell}$, for graph $\mathcal{G}_5$. All magenta crosses are below or on the 45° line.

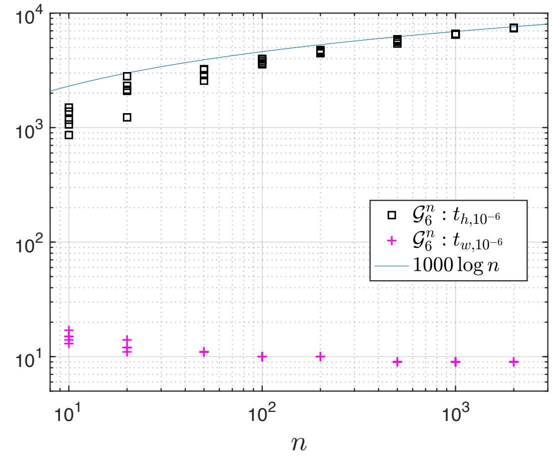

Fig. 16. Simulations of the convergence times $t_{h,10^{-6}}$ (black squares) and $t_{w,10^{-6}}$ (magenta crosses) for graphs of the family $\mathcal{G}_6^n$. There are 5 simulations for every n in $\{10, 20, 50, 100, 200, 500, 1000, 2000\}$. The solid line represents the trend $1000 \log n$.

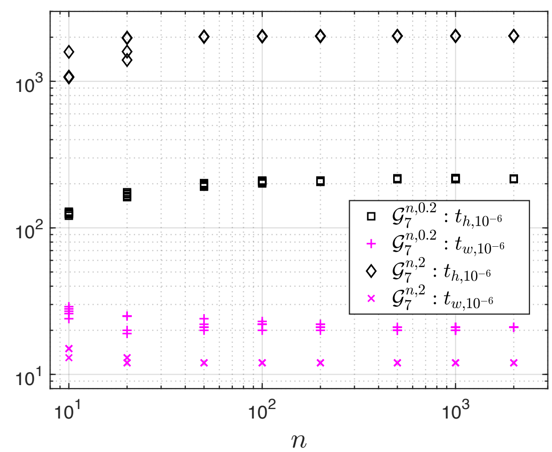

Fig. 17. Simulations of the convergence times $t_{h,10^{-6}}$ and $t_{w,10^{-6}}$ for graphs of the family $\mathcal{G}_7^{n,c}$. There are 5 simulations for each pair of (n, c) in $\{10, 20, 50, 100, 200, 500, 1000, 2000\} \times \{0.2, 2\}$. Black squares and magenta crosses are used for $t_{h,10^{-6}}$ and $t_{w,10^{-6}}$ of $\mathcal{G}_7^{n,0.2}$, respectively; black diamonds and magenta x-marks for $t_{h,10^{-6}}$ and $t_{w,10^{-6}}$ of $\mathcal{G}_7^{n,2}$, respectively.

with m the number of edges in the subgraph $\mathcal{G}[\{1, \dots, n\}]$.

As in Section 5.2, the simulations have been performed on connected graphs $\mathcal{G} = (I, E, C)$ where the field node is connected to every other node, the matrix C is symmetric and the convergence is guaranteed by Theorem 4. We denote with $\mathcal{G}_6^n$ any graph with node set $I = \{\mathfrak{f}, 1, 2, \dots, n\}$ and edge set E such that the subgraph $\mathcal{G}_6^n[\{1, \dots, n\}]$ is a connected realization of a *Erdős-Rényi* random graph with edge probability $p(n) = 1.3 \log n / n$. The entries of C follow from (22). We denote with $\mathcal{G}_7^{n,c}$, for $c \geq 0$, any graph obtained removing edges $\{i, j\}$ with $i, j \neq \mathfrak{f}$ from a $\mathcal{G}_6^n$-kind graph, so that the subgraph $\mathcal{G}_7^{n,c}[\{1, \dots, n\}]$ remains connected and has $n - 1 + cn$ edges. The matrix C is adapted accordingly.

Figure 16 shows the values of $t_{h,10^{-6}}$ and $t_{w,10^{-6}}$ for several graphs of the family $\mathcal{G}_6^n$. The values of $t_{w,10^{-6}}$ seem to slowly decrease with n and settle to 10 while those of $t_{h,10^{-6}}$ seem to concentrate and follow a trend like $1000 \log n$.

Figure 17 shows the values of $t_{h,10^{-6}}$ and $t_{w,10^{-6}}$ for several graphs of the family $\mathcal{G}_7^{n,c}$ with $c \in \{0.2, 2\}$. All times seem to concentrate and converge in n to precise values. Interestingly, the values of $t_{h,10^{-6}}$ for $c = 2$ are about ten times larger than those for $c = 0.2$ on corresponding n.

These simulations show that the convergence time of the MPA, measured by $t_{h,10^{-6}}$, has a good scaling with respect to the size n of the graph, with a moderate increase for *Erdős-Rényi* topologies, to be related with the abundance of cycles.

6 Conclusion: Open Problems

In this paper we studied the harmonic influence of nodes in a diffusion process on a graph and a message passing algorithm, originally proposed in [6] to compute an approximation of the harmonic influence. As our main contribution, we proved the convergence of the algorithm on any undirected graph, provided the Laplacian of the graph is symmetric. Simulations suggest that this assumption can be relaxed to a milder assumption of reciprocity in the interactions between the nodes: future work could focus on proving such conjecture. Our analysis is based on the concept of message digraph, which describes the relations between messages and allows us to apply suitable tools from linear algebra: this approach could be useful to analyze other message passing algorithms.

Further work could also focus on rigorously evaluating the error and the convergence time of the algorithm. Our simulations on random graphs show a very good scalability, where the convergence time seems not to depend on n but only on the number of edges m. This dependence is likely due to the adverse effects of cycles on the performance of message passing. This promising insight is confirmed by a mean-field analysis for k-regular graphs, i.e. graphs where every non-field node has the same degree k, where the convergence time depends on k only [31]. Based on these observations, we are lead to conjecture that, at least for a large class of (random) graphs, the typical convergence time of the algorithm be $O(m/n)$, where m is the number of edges.

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